HURWITZ ACTION ON TUPLES OF EUCLIDEAN REFLECTIONS

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This note was prompted by the reading of [4], which purports to show that if an n-tuple of Euclidean reflections has a finite orbit under the Hurwitz action of the braid group, then the generated group is finite. I noticed that the proof given is fatally flawed [1]; however, using the argument of Vinberg given in [3], I found a short (hopefully correct) proof which at the same time considerably simplifies the computational argument given in [3]. This is what I expound below. I first recall all the necessary notation and assumptions, expounding some facts in slightly more generality than necessary.

0.1. **Hurwitz action.**

Definition. *Given a group G, we call Hurwitz action the action of the n-strand braid group B_n with standard generators σ_i on G^n given by*

$$\sigma_i(s_1,\dots,s_n)=(s_1,\dots,s_{i-1},s_{i+1},s_i^{s_{i+1}},s_{i+2},\dots,s_n).$$

The inverse is given by $\sigma_i^{-1}(s_1,\dots,s_n)=(s_1,\dots,s_{i-1},{}^{s_i}s_{i+1},s_i,s_{i+2},\dots,s_n)$. Here a^b is $b^{-1}ab$ and ${}^b a$ is bab^{-1}.

This action preserves the product of the n-tuple. We need to repeat some remarks in [4]. By decreasing induction on i one sees that $\sigma_i\dots\sigma_n(s_1,\dots,s_n)=(s_1,\dots,s_{i-1},s_n,s_i^{s_n},\dots,s_{n-1}^{s_n})$. In particular if $\gamma=\sigma_1\dots\sigma_{n-1}$ we get $\gamma(s_1,\dots,s_n)=(s_n,s_1,\dots,s_{n-1})^{s_n}$ whence, if $c=s_1\dots s_n$, we get that $\gamma^n(s_1,\dots,s_n)=(s_1,\dots,s_n)^c$.

We also deduce that given any subsequence $(i_1,\dots,i_k)$ of $(1,\dots,n)$, there exists an element of the Hurwitz orbit of $(s_1,\dots,s_n)$ which begins by $(s_{i_1},\dots,s_{i_k})$.

Assume now that the Hurwitz orbit of $(s_1,\dots,s_n)$ is finite. Then some power of γ fixes $(s_1,\dots,s_n)$, thus some power of c is central in the subgroup generated by the s_i. Similarly, by looking at the action of $\sigma_1\dots\sigma_{k-1}$ on an element of the orbit beginning by $(s_{i_1},\dots,s_{i_k})$ we get that for any subsequence $(i_1,\dots,i_k)$ of $(1,\dots,n)$ there exists a power of $s_{i_1}\dots s_{i_k}$ central in the subgroup generated by $(s_{i_1},\dots,s_{i_k})$.

0.2. **Reflections.** Let V be a vector space on some subfield K of $\mathbb{C}$. We call *complex reflection* a finite order element $s\in\mathrm{GL}(V)$ whose fixed points are a hyperplane. If ζ (a root of unity) is the unique non-trivial eigenvalue of s, the action of s can be written $s(x)=x-\check r(x)r$ where $r\in V$ and $\check r$ is an element of the dual of V satisfying

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[1] The problem is in proposition 2.3, which is essential to the main theorem (1.1) of the paper. The argument given there is basically that if a Coxeter group has a reflection representation where the image of the Coxeter element is of finite order, then the image of that representation is finite. However this is false: the Cartan matrix $\left(\begin{smallmatrix}2&-1&-1\\-1&2&-l\\-1&-l&2\end{smallmatrix}\right)$ where $l^2+l=\sqrt{2}$ defines Euclidean reflections which give a representation of an infinite rank 3 Coxeter group, such that the image of the Coxeter group is infinite but the image of the Coxeter element is of order 8 (personal communication of F.Zara).

$\check{r}(r) = 1 - \zeta$. These elements are unique up to multiplying r by a scalar and $\check{r}$ by the inverse scalar. We say that r (resp. $\check{r}$) is a root (resp. coroot) associated to s.

0.3. **Cartan Matrix.** If $(s_1, \ldots, s_n)$ is a tuple of complex reflections and if $r_i, \check{r}_i$ are corresponding roots and coroots, we call *Cartan matrix* the matrix $C = \{\check{r}_i(r_j)\}_{i,j}$. This matrix is unique up to conjugating by a diagonal matrix. Conversely, a class modulo the action of diagonal matrices of Cartan matrices is an invariant of the $\mathrm{GL}(V)$-conjugacy class of the tuple. It determines this class if it is invertible and $n = \dim V$. Indeed, this implies that the r_i form a basis of V; and in this basis the matrix s_i differs from the identity matrix only on the i-th line, where the opposed of the i-th line of C has been added; thus C determines the s_i.

If C can be chosen Hermitian (resp. symmetric), such a choice is then unique up to conjugating by a diagonal matrix of norm 1 elements of K (resp. of signs).

If C is Hermitian (which implies that the s_i are of order 2), then the sesquilinear form given by ${}^tC$ is invariant by the s_i (if the s_i are not of order 2, but the matrix obtained by replacing all elements on the diagonal of ${}^tC$ by 2's is Hermitian, then the latter matrix defines a sesquilinear form invariant by the s_i).

0.4. **Coxeter element.** We keep the notation as above and we assume that the r_i form a basis of V. We recall a result of [2] on the "Coxeter" element $c = s_1 \ldots s_k$. If we write $C = U + V$ where U is upper triangular unipotent and where V is lower triangular (with diagonal terms $-\zeta_i$, thus V is also unipotent when s_i are of order 2), then the matrix of c in the r_i basis is $-U^{-1}V$ (to see this write it as $Us_1 \ldots s_n = -V$ and look at partial products in the left-hand side starting from the left). As U is of determinant 1, we deduce that $\chi(c) = \det(xI + U^{-1}V) = \det(xU + V)$ where $\chi(c)$ denotes the characteristic polynomial ; in particular $\det(C) = \chi(c) \mid_{x=1}$; one also gets that the fix-point set of c is the kernel of C, equal to the intersection of the reflecting hyperplanes.

0.5. **The main theorem.** The next theorem implies the statement given in [4] ([4, 1.1] considers Euclidean reflections with the r_i linearly independent; if the r_i are chosen of the same length this implies that C is symmetric, and as C is then the Gram matrix of the r_i it is invertible):

Theorem. *Let $(s_1, \ldots, s_n)$ be a tuple of reflections in $GL(\mathbb{R}^n)$ which have an associated Cartan matrix symmetric and invertible. Assume in addition that the Hurwitz orbit of the tuple is finite. Then the group generated by the s_i is finite.*

Proof. In the next paragraph, we just need that $(s_1, \ldots, s_n)$ is a tuple of complex reflections with a finite Hurwitz orbit and with the r_i a basis of V.

A straightforward computation shows than an element of $\mathrm{GL}(V)$ commutes to the s_i if and only if it acts as a scalar on the subspaces generated by $\{r_i\}_{i \in I}$ where I is a block of C (i.e., a connected component of the graph with vertices $\{1, \ldots, n\}$ and edges (i, j) for each pair such that either $C_{i,j}$ or $C_{j,i}$ is not zero). The finiteness of the Hurwitz orbit implies that for any subsequence $(i_1, \ldots, i_k)$ of $(1, \ldots, n)$, there exists a power of $s_{i_1} \ldots s_{i_k}$ which commutes to $s_{i_1}, \ldots, s_{i_k}$. This power acts thus as a scalar on each subspace generated by the r_{i_j} in a block of the submatrix of C determined by $(i_1, \ldots, i_k)$. As the determinant of each s_{i_j} on this subspace is a root of unity, the scalar must be a root of unity. Thus, the restriction of each $s_{i_1} \ldots s_{i_k}$ to the subspace $< r_{i_1}, \ldots, r_{i_k} >$ generated by the r_{i_j} is of finite order.

We use from now on all the assumptions of the theorem. Thus the s_i are order 2 elements of $O(C)$, the orthogonal group of the quadratic form defined by C.

Also, $\chi(c)$ is a polynomial with real coefficients. As c is of finite order, any real root of $\chi(c)$ is 1 or -1. This implies that $\chi(c)\mid_{x=1}$ is a nonnegative real number, and thus $\det C$ also. The same holds for any principal minor of C, since such a minor is $\chi(c')\mid_{x=1}$ where c' is the restriction of some $s_{i_1}\ldots s_{i_k}$ to $< r_{i_1},\ldots,r_{i_k} >$. The quadratic form defined by C is thus positive, and as $\det C \neq 0$ it is positive definite (cf. [1, §7, exercice 2]).

We now digress about the Cartan matrix of two reflections s_1 et s_2. Such a matrix is of the form $\begin{pmatrix} 2 & a \\ b & 2 \end{pmatrix}$. If $a=0$ and $b\neq 0$ or $a\neq 0$ and $b=0$ then s_1s_2 is of infinite order. Otherwise, the number ab is a complete invariant of the conjugacy class of (s_1,s_2) restricted to $< r_1,r_2 >$, and s_1s_2 restricted to this subspace is of finite order m if and only if there exists k prime to m such that $ab = 4\cos^2 k\pi/m$.

Since C is symmetric and since the restriction of s_is_j to $< r_i,r_j >$ is of finite order, there exists prime integer pairs $(k_{i,j},m_{i,j})$ such that $C_{i,j} = \pm 2\cos k_{i,j}\pi/m_{i,j}$. If K is the cyclotomic subfield containing the $\mathrm{lcm}(2m_{i,j})$-th roots of unity, and if $\mathcal{O}$ is the ring of integers of K, we get that all coefficients of C lie in $\mathcal{O}$. It follows, if G is the group generated by the s_i, that in the r_i basis we have $G \subset \mathrm{GL}(\mathcal{O}^n)$.

We now apply Vinberg's argument as in [3, 1.4.2]. Let $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$. Then $\sigma(C)$ is again positive definite: all arguments used to prove that C is positive definite still apply for $\sigma(C)$: it is real, symmetric, invertible and the Hurwitz orbit of $(\sigma(s_1),\ldots,\sigma(s_n))$ is still finite. Since $G \subset O(C)$, which is compact, the entries of the elements of G in the r_i basis are of bounded norm. Since $O(\sigma(C))$ is also compact for any $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$, we get that entries of elements of G are elements of $\mathcal{O}$ all of whose complex conjugates have a bounded norm. There is a finite number of such elements, so G is finite. □

References

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